Quadratic expression that produces only non-square integer and some properties of a Prime, twin primes & wings

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ABSTRACT: A quadratic expression of x always produces a square integer when the expression itself is algebraically a square expression and in this case its discriminate is always zero. But for a quadratic expression if discriminate is not equal to zero we cannot say that it will never produce a square integer. Now the question is whether or not there exists any quadratic expression which will never produce a square integer. If exists what are the boundary conditions? This paper contains two kinds of quadratic expressions that always produce a non-square integer of an even and odd numbers. It also contains some divisibility properties of a prime and twin prime mainly based on Wilson's theorem in Number Theory. An attempt has also been made to prove the famous conjecture that a square of even integer when added with one produces infinitely many primes.

Key words: Mono-center, Mono-factor, Mono-wing, Q-absorbed polynomials.

Introduction:

Once again it is felt necessary to review the theory of wings and its fundamental properties before we proceed further to establish the quadratic functions that can't produce any square integers & to explore some properties of prime and twin primes.

In a mixed combination of odd-even integers the expression $\alpha^2 \pm \beta^2$ can be said as positive or negative wing where α, β are the elements of the wing. If $(\alpha, \beta) = 1$ then the wing is called a prime wing and if $(\alpha, \beta) > 1$ it is composite wing. A prime wing therefore, may indicate a prime number or a composite number both whereas a composite wing is always for a composite number.

Basically there exists two kinds of prime numbers, 1st kind and 2nd kind according as $4x - 1$ form & $4x + 1$ form respectively. 1st kind prime cannot be expressed as $\alpha^2 + \beta^2$ whereas 2nd kind prime is always expressible as $\alpha^2 + \beta^2$ where obviously $(\alpha, \beta) = 1$. Any 2nd kind prime irrespective of its exponent has a single positive prime wing. All the composite numbers that are capable of forming this positive wing contain the prime factors of 2nd kind only. Total number of prime wings (positive or negative) of a composite number having n prime factors is 2^{n-1} with different wing lengths or elementary gap $(\alpha - \beta)$ & where all the produced elements (2^n nos.) are different. Any wing of the form $(2x)^2 \pm 1$ can be said as positive or negative mono-wing. All other wings are general wings. Any odd composite integer is of 1st kind or 2nd kind according as it is $4x - 1$ form or $4x + 1$ form. If it is of 1st kind it must contain at least one 1st kind prime and sum of their all exponents must be odd. If it is of 2nd kind sum of exponents of all 1st kind prime must be even or zero i.e. no 1st kind prime is present and then it can be said as purely 2nd kind composite number.

Product of two positive or negative wings produces two equal valued positive or negative wings by N_s or N_d operation respectively.

We can divide all the twin primes into two categories. Twin primes of 1st kind are those where 1st kind prime < 2nd kind prime and 2nd kind twin primes includes 2nd kind prime < 1st kind prime. Property wise these two kind of twin primes have some differences.

One notation $\uparrow(N)_x$ has been used here to indicate the exponent of prime factor of x in a number N.

1. Quadratic function that fails to produce square integer

1.1 Product of two positive or negative prime wings having a common element d cannot produce another wing of same element d.

1.2 $(x^2 \pm d^2)(y^2 \pm d^2) \mp d^2 \neq I^2$ for $(y, d) = 1$ & $(x, d) \neq d$

By Ns-operation we have, $(x^2 + d^2)(y^2 + d^2) = (xy \pm d^2)^2 + \{d(x - y)\}^2$
Here, if $d(x - y) = d \Rightarrow x \pm y = 1$ which is not possible for even-odd combination of elements of a wing.
If $xy \pm d^2 = d$ produced wing is a composite wing. But $x^2 + d^2$ & $y^2 + d^2$ both being prime wings, its product cannot be a composite wing.
Hence, $(x^2 + d^2)(y^2 + d^2) - d^2 \neq I^2$ for $(y, d) = 1$ and $x \neq \lambda d$ when $d > 1$.
e.g. if $d = 1$, $y = 4$; $17x^2 + 16 \neq I^2$ where x is even. $\Rightarrow 68x^2 + 16$ i.e. $17x^2 + 4 \neq I^2$ for all integers of x
if $d = 3$, $y = 8$; $\{(2x)^2 + 9\}.73 - 9 \neq I^2 \Rightarrow 73x^2 + 162 \neq I^2$ for $x \neq 3\lambda$.
If $d = 4$, $y = 7$; $65x^2 + 1024 \neq I^2$ where x is odd. $\Rightarrow 65(2x - 1)^2 + 1024 \neq I^2$ i.e. $260x^2 - 260x + 1089 \neq I_o^2$ for all integers of x as $2x - 1 \neq 4\lambda$ Similarly, for $d = 6$, $y = 1$; $148x^2 - 148x + 1333 \neq I_o^2$ for all integers of x as $2x - 1 \neq 6\lambda$.
Again by N_d-operation we have, $(x^2 - d^2)(y^2 - d^2) = (xy \pm d^2)^2 - \{d(x - y)\}^2$ where applying same logic we can say, $(x^2 - d^2)(y^2 - d^2) + d^2 \neq I^2$
e.g. for $d = 1$, $y = 4$; $(x^2 - 1).15 + 1$ i.e. $15x^2 - 14 \neq I^2$ where x is even $\Rightarrow 60x^2 - 14 \neq I^2$ for all integers of x.
For $d = 3$, $y = 8$; $\{(2x)^2 - 9\}.75 + 9 \neq I^2 \Rightarrow 300x^2 - 666 \neq I^2$ for $x \neq 3\lambda$
If $d = 6$, $y = 7$; $\{(2x - 1)^2 - 36\}.13 + 36 \neq I^2 \Rightarrow 52x^2 - 52x - 419 \neq I^2$ for all integers of x as $2x - 1 \neq 6\lambda$
$\Rightarrow$ The quadratic functions that can't produce square of odd integers are always in the form of $f(x) = ax^2 - ax + c \neq I_o^2$ where a is in the form of 4(odd integer) $\Rightarrow f(1) = |c| \neq I_o^2$

Corollary: If one of the produced wings (positive or negative) of M and that of N (positive or negative) have a common element like $a^2 \pm \alpha^2$ & $b^2 \pm \alpha^2$ then product of any produced wing of M & that of N cannot produce any wing having same element α.

2. For a quadratic function $f(x) = ax^2 + bx + c$ if $f(x_0) = \omega_o^2$ (ω_o denotes an odd integer) then f(x) will produce infinitely many square of odd integers for infinitely many integers of x.

It is quite justified to assume that a quadratic function which is not of non-square integer nature, produces at least one square integer i.e. $f(x_0) = ax_0^2 + bx_0 + c = \omega_o^2 \Rightarrow f(x_0 + \alpha) = a\alpha^2 + (2ax_0 + b)\alpha + \omega_o^2 = \varphi(\alpha)$ (say).
Obviously, $\varphi(\alpha)$ is not of non-square nature due to presence of ω_o^2 and it will produce a square odd integer at least for once. $\Rightarrow f(x_1) = \omega_1^2$ i.e. $ax_1^2 + bx_1 + c = \omega_1^2$ and following the same logic we can get infinite nos. of square odd integers i.e. $f(x_\infty) = \omega_\infty^2$
Note: For a quadratic function $f(x) = 4\lambda x^2 - 4\lambda x \pm c$ where λ is a positive prime wing & c is a non-square odd integer we cannot say that f(x) produces non-square integers but if any one breaks the conditions f(x) will produce infinitely many square of odd integers.

3. Biquadrate function that produces non-square integers

We have $(x^2 \pm d^2)(y^2 \pm d^2) \mp d^2 \neq I^2$ when d is even and x is odd.
Replacing x by $2x - 1$, $4(y^2 \pm d^2)x^2 - 4(y^2 \pm d^2)x + \{y^2 \pm d^2(y^2 \pm d^2)\} \neq I^2 \Rightarrow \{y^2 \pm d^2(y^2 \pm d^2)\} \neq I^2$ at $x = 1$
$\Rightarrow d^4 \pm y^2d^2 + y^2 \neq I^2$. Now replacing d by 2x & y by $2\lambda - 1$, $16x^4 \pm 4(2\lambda - 1)^2x^2 + (2\lambda - 1)^2 \neq I_o^2$ where $(2\lambda - 1, x) = 1$.

4. Quadratic function that fails to produce square of an even integer.

4.1 For $\alpha > \beta$, $2(\alpha^2 + \beta^2)x^2 + 2(\alpha - \beta)(\alpha^2 + \beta^2)x + (\alpha^2 + \beta^2)(\alpha - \beta)^2 - \lambda_i^2 \neq I_e^2$ where $\lambda_i = 1, 3, 5, \ldots\ldots < (\alpha - \beta)$

Product of any two positive wings like $\{(x + d)^2 + x^2\}\{(y + d)^2 + y^2\}$ produces two wings having lowest element $d|x - y|$ where obviously d is odd. So it cannot produce a mono-wing unless $d = 1$ & $|x - y| = 1$

$\Rightarrow (\alpha^2 + \beta^2)\{(x + \alpha - \beta)^2 + x^2\} - 1 \neq I_e^2$ for $\alpha > \beta$ and where $\alpha - \beta$ & $x - \beta$ both $\neq 1$ at a time.
$\Rightarrow 2(\alpha^2 + \beta^2)x^2 + 2(\alpha - \beta)(\alpha^2 + \beta^2)x + (\alpha^2 + \beta^2)(\alpha - \beta)^2 - 1 \neq I_e^2$ for $\alpha > \beta$ & where $x \neq \beta + 1$ if $\alpha - \beta = 1$.
In a particular case where $x = 1$, $\alpha - \beta = 1$; $5(\alpha^2 + \beta^2) - 1 \neq I_e^2 \Rightarrow 10\alpha^2 + 10\alpha + 4 \neq I_e^2$
In general, for $\alpha > \beta$, $2(\alpha^2 + \beta^2)x^2 + 2(\alpha - \beta)(\alpha^2 + \beta^2)x + (\alpha^2 + \beta^2)(\alpha - \beta)^2 - \lambda_i^2 \neq I_e^2$ where $\lambda_i = 1, 3, 5, \ldots\ldots < (\alpha - \beta)$
Nature of this type of function:
Coefficients of x^2, x & free term all are positive & even.
Coefficient of x^2 is in the form of 2(positive wing) & that of x is in the form of 2(an odd integer having a factor of positive wing)
Free term i.e. $(\alpha^2 + \beta^2)(\alpha - \beta)^2 - \lambda_i^2 \neq I_e^2$ for $\lambda_i = 1, 3, 5, \ldots\ldots < (\alpha - \beta)$
Say $\alpha - \beta$ contains a 1st kind prime & $(\alpha^2 + \beta^2)(\alpha - \beta)^2 = \lambda_i^2 + I_e^2$
As $(\alpha - \beta)^2$ fails to produce a positive wing all the produced wings of left hand side are composite with a common factor $(\alpha - \beta)$ in between the elements.
$\Rightarrow (\lambda_i, I_e) = \alpha - \beta$ which is impossible as $\lambda_i = 1, 3, 5, \ldots\ldots < (\alpha - \beta)$
If $\alpha - \beta$ is of purely 2nd kind, prime or composite, $(\alpha - \beta)^2$ will produce several positive wings but in none of the wings difference of the elements will be $(\alpha - \beta)^2$
So, $(\alpha^2 + \beta^2)(\alpha - \beta)^2 \neq \lambda_i^2 + I_e^2$ whatever may be the nature of $(\alpha - \beta)$

Here also applying the same logic of point-2, we can say, for a quadratic function $f(x) = ax^2 + bx + c$ if $f(x_0) = \omega_e^2$ (ω_e denotes an even integer) then f(x) will produce infinitely many square of odd integers for infinitely many integers of x.

Similarly, by N_d-operation for the product of two negative wings we can say, $(\alpha^2 - \beta^2)\{(x + \alpha - \beta)^2 - x^2\} + \lambda_i^2 \neq I_e^2$
$\Rightarrow 2(\alpha - \beta)^2(\alpha + \beta)x + (\alpha - \beta)^3(\alpha + \beta) + \lambda_i^2 \neq I_e^2$ where $\lambda_i = 1, 3, 5, \ldots\ldots < (\alpha - \beta)$

5. If $p_1 = (x + d)^2 + x^2$ & $p_2 = (y + d)^2 + y^2$ represent two primes of same prime length d where $d|x - y| = \lambda$ (say), then $p_1p_2 - \alpha_i^2 \neq I^2$ for $\alpha_i = 1, 2, 3, \ldots\ldots, \lambda - 1$

As p_1 & p_2 are primes they have no other wings but $(x + d)^2 + x^2$ & $(y + d)^2 + y^2$ where d is obviously an odd integer for mixed combination of elements.
$\Rightarrow p_1p_2 = (2xy + dx + dy + d^2)^2 + \{d(x - y)\}^2 = (dx + dy + d^2)^2 + (2xy + dx + dy)^2$ by N_s-operation.
Obviously, lowest element is $d|x - y| = \lambda$ say.
$\Rightarrow p_1p_2 - \alpha_i^2$ cannot be a square integer for $\alpha_i = 1, 2, 3, \ldots\ldots, \lambda - 1$.
e.g. $101.353 = (10^2 + 1)(17^2 + 8^2)$ where $d|x - y| = 9.7 = 63 \Rightarrow 101.353 - \alpha_i^2 \neq I^2$ where $\alpha_i = 1, 2, 3, \ldots\ldots, 62$

Note: d can be said as wing length and for a particular value of d, all primes satisfy c of a 1st kind N-equation having $k = d^2$.

6. Wilson's theorem in generalized form and twin prime.

According to Wilson's Theorem if p is prime $p|(p - 1)! + 1$
$\Rightarrow p|(p - 1).(p - 2)! + 1 \Rightarrow p|(p - 2)! - 1 \Rightarrow p|(p - 2).(p - 3)! - 1 \Rightarrow p|2!(p - 3)! + 1 \Rightarrow p|3!(p - 4)! - 1$ and so on.
In general, $p|(\lambda - 1)!(p - \lambda)! + (-1)^{\lambda - 1}$ ……….. (A1)
If p + 2 is also prime, replacing p by p + 2 we have $(p + 2)|(\lambda - 1)!(p + 2 - \lambda)! + (-1)^{\lambda - 1}$
Now, replacing λ by $\lambda + 2$, we have $(p + 2)|(\lambda + 1)!(p - \lambda)! + (-1)^{\lambda + 1}$ ……….. (B1)
e.g. for $\lambda = 1$ from A1 we get $p|(p - 1)! + 1$ …… (A2) & from B1 we get $(p + 2)|2.(p - 1)! + 1$ ……. (B2)
for $\lambda = 2$ from A1 we get $p|(p - 2)! - 1$ …… (A3) & from B1 we get $(p + 2)|6.(p - 2)! - 1$ ……. (B3) & so on for twin prime (p, p + 2)
As prime number exists infinitely so for p is prime all A_i are bound to be satisfied. If B_i is also satisfied then (p, p + 2) is bound to be twin prime as W. Theorem is also true in converse way.
Twin primes are always in combination with 1st kind (4x – 1 form) and 2nd kind (4x + 1) form.

If the 2nd kind prime is 2q + 1 (obviously q is even) then obviously $(2q + 1) | (q!)^2 + 1$ and the other prime if it is greater, then $(2q + 3) | \{(q + 1)!\}^2 - 1$ and if it is smaller $(2q - 1) | \{(q - 1)!\}^2 - 1$.
e.g. for the twin prime (59, 61) where 61 is the 2nd kind, $61 | (30!)^2 + 1$ and $59 | (29!)^2 - 1$
But for twin prime (29, 31) where 29 is of 2nd kind $29 | (14!)^2 + 1$ and $31 | (15!)^2 - 1$.

Corollary: For an even integer q if 2q + 1 belongs to prime $(q!)^2 + 1$ belongs to composite containing the factor 2q +1. Obviously, if $(q!)^2 + 1$ is prime 2q + 1 is composite but it cannot contain the factor $(q!)^2 + 1$. When q is in the form of $2q^2$ we can say $(2q)^2 + 1$ & $\{(2q^2)!\}^2 + 1$ both cannot be prime.

7.1 For a second kind prime 2q + 1 if 2q + 3 is prime then q + 1 must be composite but reverse is not true i.e. if q + 1 is composite 2q + 3 may be composite also.
7.2 For a second kind prime 2q + 1 if q + 1 is prime then 2q + 3 must be composite but reverse is not true i.e. if 2q + 3 is composite q + 1 may be composite also.

Case I, where 1st kind prime > 2nd kind.

Say, 2q + 1 is a 2nd kind prime & $\Rightarrow (2q + 1) | (q!)^2 + 1$. Say, $(q!)^2 + 1 = \mu(2q + 1)$ …….. (C)
Now, if $(2q + 3) | \{(q + 1)!\}^2 - 1$ then (2q + 1, 2q + 3) must be twin prime.
Say, $(2q + 3) | \{(q + 1)!\}^2 - 1 \Rightarrow$ either $(2q + 3) | (q + 1)! - 1$ or, $(2q + 3) | (q + 1)! + 1$
Considering the 1st one, say $(q + 1)! - 1 = \omega(2q + 3)$ ………. (D)
As per (C), $(q + 1)!q! + (q + 1) = \mu(2q + 1)(q + 1)$ or, $(q + 1)! = \mu(2q + 1)(q + 1)/q! - (q + 1)/q!$
$\Rightarrow (q + 1)! = \mu(2q + 3)(q + 1)/q! - (q + 1)/q! - 2\mu(q + 1)/q! \Rightarrow (q + 1)! = \mu(2q + 3)(q + 1)/q! - (2\mu + 1)(q + 1)/q!$
$\Rightarrow (q + 1)! = \mu(2q + 3)(q + 1)/q! + 1 - \{1+ (2\mu + 1)(q + 1)/q!\}$
& it can match with the form of (D) when & only when $\{1+ (2\mu + 1)(q + 1)/q!\} = \gamma(2q + 3)(q + 1)/q!$
$\Rightarrow \gamma(2q + 3) - \delta = q!/(q + 1)$ which has integer solutions provided q + 1 is composite & for which 2q + 3 is prime.
On the other hand if q + 1 is prime 2q + 3 is composite. In both the cases reverse is not true. We cannot ignore the cases where with respect to 2nd kind prime 2q + 1, 2q + 3 & q + 1 both are composites.
Similarly, if $(2q + 3) | (q + 1)! + 1$ for 2nd one, we will get the same type of relation where same logic is applicable.

Case II, where 1st kind prime < 2nd kind

Here, $(q!)^2 + 1 = \mu(2q + 1)$ …….. (C)
Now, if $(2q - 1) | \{(q - 1)!\}^2 - 1$ then (2q – 1, 2q + 1) must be twin prime.
Say, $(2q - 1) | \{(q - 1)!\}^2 - 1 \Rightarrow$ either $(2q - 1) | (q - 1)! - 1$ or, $(2q - 1) | (q - 1)! + 1$
Considering the 1st one, say $(q - 1)! - 1 = \omega(2q - 1)$ ………. (E)
Now, as per (C), $(q - 1)! = \mu(2q + 1)/(q.q!) - 1/(q.q!) \Rightarrow (q - 1)! = \mu(2q - 1)/(q.q!) + (2\mu - 1)/(q.q!)$
$\Rightarrow (q - 1)! - 1 = \mu(2q - 1)/(q.q!) + \{(2\mu - 1)/(q.q!) - 1\}$
And it will be matching with (E) when & only when $\{(2\mu - 1)/(q.q!) - 1\} = \gamma(2q - 1)/(q.q!)$
$\Rightarrow (2\mu - 1) - \gamma(2q - 1) = q.q!$ which has integer solutions without any conditional phenomenon.
Similarly for other case we will get same type of relation.
So, for 1st kind twin prime $(2q \pm 1)$, q + 1 may be composite or may be prime.

Corollary: If $(2^n q + 1, 2^n q + 3)$ represents a twin prime then $(2^{n \pm 1}q + 1, 2^{n \pm 1}q + 3)$ or $(2^{n-1}q \pm 1)$ cannot represent again a twin prime.

8.1 Condition of divisibility by a 1st kind prime 2q – 1 in a different form

We have $(2q - 1) | (q - 1)! - 1$ or $(2q - 1) | (q - 1)! + 1$ where obviously q is even.
For 1st case, $(2q - 1) | (2q - 2).(q - 2)! - 2 \Rightarrow (2q - 1) | 1.(q - 2)! + 2 \Rightarrow (2q - 1) | 1.(2q - 4).(q - 3)! + 4$
$\Rightarrow (2q - 1) | 1.3.(q - 3)! - 4 \Rightarrow (2q - 1) | 1.3.(2q - 6).(q - 4)! - 8 \Rightarrow (2q - 1) | 1.3.5.(q - 4)! + 8$ & so on
In general, $(2q - 1) | (1.3.5\ldots.. \text{ n terms}).(q - n - 1)! - (-2)^n$

& for $n = q - 1$, $(2q - 1)|(1.3.5\ldots\ldots q - 1 \text{ terms}) + 2^{q-1}$
Similarly, for other case $(2q - 1)|\ (1.3.5\ldots\ldots q - 1 \text{ terms}) - 2^{q-1}$
Clubbing both we can say, $(2q - 1)|\ (1.3.5\ldots\ldots q - 1 \text{ terms})^2 - (2^{q-1})^2$

8.2 Condition of divisibility by a 2nd kind prime 2q + 1 in a different form

We have $(2q + 1)|(q!)^2 + 1 \Rightarrow (2q + 1)|2q.(q - 1)!q! + 2 \Rightarrow (2q + 1)|(2q + 1 - 1) .(q - 1)!q! + 2$
$\Rightarrow (2q + 1)|1.(q - 1)!q! + (-2)^1 \Rightarrow (2q + 1)|1.(2q - 2).(q - 2)!.q! + 2. (-2)^1 \Rightarrow (2q + 1)|1.3.(q - 2)!q! + (-2)^2$
$\Rightarrow(2q + 1)|1.3.(2q - 4).(q - 3)!q! + 2(-2)^2 \Rightarrow (2q + 1)|1.3.5.(q - 3)!q! + (-2)^3$ & so on.
In this way if we proceed for both q! finally we will get
$(2q + 1)|\{(1.3.5\ldots\ldots m \text{ terms}).(q - m)!\}\{(1.3.5\ldots\ldots n \text{ terms}).(q - n)!\} + (-2)^{m+n}$
& for $m = n = q$, $(2q + 1)|\{(1.3.5\ldots\ldots q \text{ terms})\}^2 + (2^q)^2$

9. For a 1st kind prime (2q – 1), $(2q - 1)|\{(q - 1)!\}^2 - 1 \Rightarrow (2q - 1)|(q - 1)! - 1$ or $(2q - 1)|(q - 1)! + 1$

When $(2q - 1)|(q - 1)! - 1$ then $(2q - 1)|q! + q - 1$ & $q! - (q - 1)! + q$ both.
When $(2q - 1)|(q - 1)! + 1$ then $(2q - 1)|q! - q + 1$ & $q! - (q - 1)! - q$ both.

For a 1st kind prime $(2q - 1)|\{(q - 1)!\}^2 - 1$ & $2q - 1$ can be written as $q^2 - (q - 1)^2$
So by N_d-operation, quotient = $[\{q.(q - 1)! \pm (q - 1)\}/(2q - 1)]^2 - [\{(q - 1).(q - 1)! \pm q\}/(2q - 1)]^2$ which gives following four possibilities.
$(2q - 1)|(q - 1)! - 1,\ q! + q - 1,\ q! - (q - 1)! + q$
$(2q - 1)|(q - 1)! + 1,\ q! + q - 1,\ q! - (q - 1)! + q$
$(2q - 1)|(q - 1)! - 1,\ q! - q + 1,\ q! - (q - 1)! - q$
$(2q - 1)|(q - 1)! + 1,\ q! - q + 1,\ q! - (q - 1)! - q$
Here 2nd & 3rd are absurd. Say, for 2nd one, if it is true then $(2q - 1)|\{q! + q - 1\} - \{q! - (q - 1)! + q\}$
$\Rightarrow (2q - 1)|(q - 1)! - 1$ But $(2q - 1)$ cannot divide $(q! \pm 1)$ both. Similarly, 3rd one can also be ruled out.
So possibility lies only in between first & fourth. Hence, it is proved.

- For N_d & N_s operation refer my earlier publication to IJSER – Aug-edition 2013 & Nov-edition 2015.

10. How a composite mono-wing is formed

Let us consider any odd integer d and its adjacent even number v where $\uparrow(v)_2 > 1$.
Say, $v_1v_2 = v$ (product of two even integers) & $d = d_1d_2$
Now, by N_s-operation $(v_1^2 + d_1^2)(v_2^2 + d_2^2) = (v_1d_2 + v_2d_1)^2 + |v_1v_2 - d_1d_2|^2 = (v_1d_2 + v_2d_1)^2 + 1^2$ which is a composite mono-wing.
Now, if $d_1 = 1$ we have $(v_1^2 + 1^2)(v_2^2 + d_2^2) = (v_1d_2 + v_2)^2 + |v_1v_2 - d_2|^2 = (v_1d_2 + v_2)^2 + 1$
$\Rightarrow (v_1^2 + 1^2)\{v_2^2 + (v_1v_2 \pm 1)^2\} = (v_1d_2 + v_2)^2 + 1$ where any one of ± is to be considered.
i.e. $MM' = M_1$ where M is a mono-wing prime or composite & M' is its mono-factor, M_1 is a composite mono-wing.
If d_1 or $d_2 \neq 1$, $(v_1^2 + d_1^2)(v_2^2 + d_2^2) = (v_1d_2 + v_2d_1)^2 + 1^2$ i.e. $GG' = M$ where G denotes general wing.
So, a composite mono-wing is formed by the product of two general wings where elements are taken from the factors of two consecutive integers for which all wings are bound to be prime wings. A particular case is the product of a mono-wing and its mono-factor of a general wing.
A particular mono-wing has infinite nos. of mono-factors to produce infinite nos. of different composite mono-wings i.e. $\{(2\alpha)^2 + 1^2\}\{(2x)^2 + (4\alpha x \pm 1)^2\} = \{2x + 2\alpha(4\alpha x \pm 1)\}^2 + 1$ where $x = 1, 2, 3, \ldots\ldots$ & $(2\alpha)^2 + 1^2$ is given.
All the theories are also applicable for negative wings following N_d-operation.
e.g. let us consider two consecutive nos. 40 & 39 where $\uparrow(40)_2 = 3$
Pair-wise all the even factors of 40 are 2, 20; 4,10; and all the factors of 39 are 1, 3, 13, 39.
Hence, so far $MM' = M$ examples are concerned they are $(2^2 + 1)(20^2 + 39^2) = 98^2 + 1$; $(4^2 + 1)(10^2 + 39^2) = 166^2 + 1$; $(10^2 + 1)(4^2 + 39^2) = 394^2 + 1$ & $(20^2 + 1)(2^2 + 39^2) = 782^2 + 1$

So far $GG^{/} = M$ examples are concerned they are $(2^2 + 3^2)(20^2 + 13^2) = 86^2 + 1$; $(2^2 + 13^2)(20^2 + 3^2) = 266^2 + 1$; $(4^2 + 3^2)(10^2 + 13^2) = 82^2 + 1$; $(4^2 + 13^2)(10^2 + 3^2) = 142^2 + 1$

The set $M_c(w_m) = \{(2^2 + 1), (4^2 + 1), (10^2 + 1), (20^2 + 1)\}$ can be said as mono-wings on product side and the set $M_c(w_c) = \{98^2 + 1$ or $166^2 + 1$ or $394^2 + 1$ or $782^2 + 1$ or $86^2 + 1$ or $266^2 +1$ or $82^2 + 1$ or $142^2 + 1\}$ can be said as composite mono-wings on produced side with respect to a mono-center $M_c(40, 39)$. The set $M_c(w_m)$ can be symbolically represented by $\{$any wing of $w_c\}^{/}$ e.g. $\{98^2 + 1\}^{/} = \{(2^2 + 1), (4^2 + 1), (10^2 + 1), (20^2 + 1)\}$

So, a group of composite mono-wings has a fixed mono-center with respect to which there exists a group of mono-products. Each mono-product is a product of a mono-wing & its mono-factor. Any mono-center must produce at least two mono-products where one mono wing is common i.e. $2^2 + 1$. With respect to a fixed mono-center the group of mono-products or mono-wings or mono-factors and group of produced composite mono-wings are unique. There cannot be a common wing in between $M_c(w_m)$ & $M_c(w_c)$. Any wing of $M_c(w_c) >$ any wing of $M_c(w_m)$. All the composite mono-wings under w_m have distinct mono-center. For a mono-center where the odd element is prime, all wings of $\{w_c\}$ are formed by M-product only.

10.1 With respect to a positive (or negative) general prime wing $N = (2\alpha)^2 + \beta^2$, $2\alpha < \beta$ there exists only one mono-wing M so that MN will again represent a mono wing subject to condition $2\alpha|\beta + 1$ or $2\alpha|\beta - 1$

Given, $N = (2\alpha)^2 + \beta^2$ Say, $M = (2x)^2 + 1 \Rightarrow MN = |4\alpha x - \beta|^2 + (2\alpha + 2x\beta)^2$ where $|4\alpha x - \beta| = 1$ i.e. $2x = \beta \pm 1$

Now, if $\beta + 1$ is in the form of 2d, $\beta - 1$ must be in the form of $2^n d$ where $n > 1$ or vice-versa. $\Rightarrow \beta \pm 1$ both cannot be equal to $4\alpha x$. Hence, any one of $\beta \pm 1$ must be even multiple of 2x.

e.g. say, $N = 50^2 + 99^2$. Here, $50|(99 + 1)$ & $= 2$. $\Rightarrow M = 2^2 + 1$ & $MN = (2^2 + 1)(50^2 + 99^2) = 248^2 + 1$

Say, $N = 50^2 + 101^2$ where $50|(101 - 1)$ & $= 2 \Rightarrow M = 2^2 + 1$ & $MN = (2^2 + 1)(50^2 + 101^2) = 252^2 + 1$

Say, $N = 12^2 + 49^2$ where $12|(49 - 1)$ & $= 4 \Rightarrow M = 4^2 + 1$ & $MN = (4^2 + 1)(12^2 + 49^2) = 208^2 + 1$

Similarly, for negative wings, $(2^2 - 1)(99^2 - 50^2) = 148^2 - 1$ & $(2^2 - 1)(101^2 - 50^2) = 152^2 - 1$

10.2 With respect to any even element 2x there exists infinitely many prime wings (positive or negative) $(2x)^2 + (2xt \pm 1)^2$ which when multiplied by a mono-wing $t^2 + 1$ will produce again a mono-wing, t being a parameter of even integer.

By Ns-operation we have one of the wings of $\{(2x)^2 + (2xt \pm 1)^2\}(t^2 + 1) = \{2x + t(2xt \pm 1)\}^2 + 1$ where t can be chosen as an even parameter.

10.3 With respect to any positive prime wing $N = (2\alpha)^2 + \beta^2$ where $2\alpha > \beta$ or a composite wing there cannot exist any mono-wing M so that $M_1N = M_2$

It is quite understood.

10.4 For any prime wing say, $w = \alpha^2 + \beta^2$, $(\alpha, \beta) = 1$; there must exist infinite nos. of prime wings w_i so that ww_i will produce a mono-wing.

As $(\alpha, \beta) = 1$, we have $\alpha x - \beta y = 1$ has infinitely many integer solutions.

$\Rightarrow (\alpha^2 + \beta^2)(x^2 + y^2) = v^2 + 1.$

10.5 If $(2\lambda)^2 + 1$ is a product of a mono-wing and its mono factor then there must exist an integer α so that $(4\alpha^2 + 1)|(\lambda \pm \alpha)$, $\lambda > \alpha$

By product of M-theory we have $\{(2\alpha)^2 + 1\}\{(2x)^2 + (4\alpha x \pm 1)^2\} = (2\lambda)^2 + 1$

$\Rightarrow 2x + 2\alpha(4\alpha x \pm 1) = 2\lambda \Rightarrow x = (\lambda \pm \alpha)/(4\alpha^2 + 1)$ where obviously $\lambda > \alpha$ & $(\lambda \pm \alpha) > (4\alpha^2 + 1) \Rightarrow 4\alpha^2 \pm \alpha + 1 < \lambda$

$\Rightarrow 4\alpha^2 + \alpha - (\lambda - 1) < 0 \Rightarrow \alpha \in (0, [\mu])$, μ is a positive root of $4x^2 - x - (\lambda - 1) = 0$ & $[\mu]$ is the greatest integer of μ.

Nos. of cases that α lies in $(0, [\mu])$ to satisfy $(4\alpha^2 + 1)|(\lambda \pm \alpha)$ are the nos. of mono-wings that $\{w_m\}$ contains.

Here, $(2x)^2 + (4\alpha x \pm 1)^2 - 1 \neq I^2 \Rightarrow 2\alpha x(2\alpha x \pm 1) + x^2 \neq I^2 \Rightarrow P_1P_2 + x^2 \neq I^2$ where P_1 & P_2 are two consecutive integers & x is any factor of them. Or, we can say, $P_1P_2 + x^{2n} \neq I^2$ if $P_1P_2x^n$ can be expressed as product of two consecutive integers $\Rightarrow$ if (P_1x^2) & P_2 are two consecutive integers then $P_1P_2 + x^2 \neq I^2$. It is also true that $P_1P_2 \neq I^2$ if they are consecutive. It therefore, establishes the following quadratic functions with fixed coefficients that fails to produce square integers.

1. $x(x \pm 1) \neq I^2 \Rightarrow x^2 \pm x \neq I^2$
2. $x(x \pm 1) + 1 \neq I^2 \Rightarrow x^2 \pm x + 1 \neq I^2$
3. $x(x \pm 1) + x^2 \neq I^2 \Rightarrow 2x^2 \pm x \neq I^2$
4. $x(x \pm 1) + (x \pm 1)^2 \neq I^2 \Rightarrow 2x^2 \pm 3x + 1 \neq I^2$

In two cases i.e. $x^2 - x + 1$ & $2x^2 - x$, $x \neq 1$ because of 0 & 1 which cannot be accepted as consecutive integers. Here, x can be replaced by any polynomial function with integer coefficients and in general
$\{f(x)\varphi(x)\}\{ f(x)\varphi(x) \pm 1\} + \{f(x) \text{ or } \varphi(x)\}^2 \neq I^2$

Note: Square free integers received by a quadratic function fully rests on the fact that $P_1P_2 + x^2 \neq I^2$ where P_1 & P_2 are two consecutive integers and x is a factor of P_1 or P_2 or 0. When x = 1 or P_1 or P_2 we receive the above four square-free quadratic functions. But when x is a part of P_1 or P_2 we receive all the square-free quadratic functions as mentioned in theorems 1 & 4.
In all cases if f(x) be a square-free quadratic function then $f\{\varphi(x)\}$ is a square-free polynomial and can be said as Q-absorbed polynomials where $\varphi(x)$ is any polynomial function of integer coefficients. Obviously, highest degree of Q-absorbed polynomial is even whose co-efficient for theorem 1 & 4, are in the form of $4dI^2$ & $2dI^2$ respectively where d is of purely 2nd kind integer.

11. Any linear expression with integer coefficients can produce infinitely many square integers.

Say, f(x) = ax + b represents a square free integer function. $\Rightarrow$ if x is replaced by any polynomial of x with integer coefficients its nature must remain unchanged. Replace, x by $ax^2 + x$ i.e. $f(ax^2 + x) = a^2x^2 + ax + b$ also represents a square-free quadratic function. Now, if $a \neq 1$, it must be matching with theorem 1 or 4. But it cannot match with theorem 1 as because $a^2 \neq a$. It also fails to match with theorem 4, as coefficient of x^2 is in the form of $2I_o$ which cannot be equated with a square integer a^2.
If a = 1, linear function is x + b which obviously can produce infinitely many square & square-free integers both

12. Any polynomial with integer coefficients & where highest degree is odd produces infinitely many square integers by magnitude.

Let us consider a polynomial of odd-degree $f(x) = ax^{2n+1} + a_1x^{2n} + a_2x^{2n-1} + \ldots\ldots..$ which always produces square-free integers. So, if x is replaced by any polynomial with integer co-efficient its nature will remain unchanged. Replace x by px^2 where p is a 1st kind prime. $\Rightarrow$ coefficient of $x^{2(2n+1)}$ of $f(px^2)$ is ap^{2n+1} which cannot be equated neither with $4dI^2$ nor $2dI^2$ as mentioned in note of theorem 10. $\Rightarrow$ $f(px^2)$ cannot represent a polynomial for square-free integers. $\Rightarrow$ f(x) cannot represent a polynomial for square-free integers. Hence proved.

13. There exists infinitely many primes of the form $u^2 + 1$ and in general with respect to a fixed odd element of a positive wing.

Every alternate even number v has the property $\uparrow(v)_2 > 1 \Rightarrow$ existence of the set $\{v, v \pm 1\}$ is infinite. Each set produces some finite nos. of composite mono-wings among which at least two are by the product of mono-wing & its mono-factor, known as M-product and others, if remains, are by the product of two general wings known as G-product. All the produced mono-wings by all infinite sets are distinct & unique. So, the existence of composite mono-wings by M-product is infinitely extended along with that of G-product as mono-center tends to infinity.

With respect to a mono-center $M_c(v, v + 1)$ say $M_c(w_m) = \{w_i, w_j\}$, i, j = 1, 2, 3, ……. where w_i represents all mono-wings of prime number & w_j represents all mono-wings of composite numbers.
Say a particular wing $w_t \in w_j \Rightarrow \{w_k\}' = \{w_i', w_j'\}$ where $w_i' \in$ prime & $w_j' \in$ composite.
Again if $w_{k'} \in w_j'$, $\{w_{k'}\}' = \{w_i'', w_j''\}$ & so on.
Now, the collection of all prime numbers $\{w_i, w_i', w_i'', \ldots\}$ can be represented by $M_C(p_i)$ which is unique with respect to a mono-center either by number of primes or by magnitude of prime.
So, as mono-center tends to infinity prime number of the form $(2u)^2 + 1$ also tends to infinity.
In general, there exists infinitely many primes with respect to a fixed odd element of a positive wing.

Note: among all $M_C(p_i)$ for different values of c, one prime is common i.e. $2^2 + 1 = 5$. So, $M_C(p_i)$ is the set of all extracted primes from a composite mono-wing.

14. A particular class of mono wings can produce infinitely many primes.

We have, $\{(2t)^2 + 1\}\{(2x)^2 + (4xt \pm 1)^2\} = \{2x + 2t(4xt \pm 1)\}^2 + 1 = (2\lambda)^2 + 1$ (say)
$\Rightarrow 2x + 2t(4xt \pm 1) = 2\lambda$ or, $4xt^2 \pm t - (\lambda - x) = 0$ (obviously, $\lambda > x$)
$\Rightarrow 4x(\lambda - x)$ must be product of two consecutive integers which is possible in infinite nos. of ways for different values of λ. One of the case may be $4x$ & $(\lambda - x)$ are consecutive i.e. $\lambda = 5x \pm 1$
If $4x$ & $(\lambda - x)$ are consecutive then $y^2 \pm y = 4x(\lambda - x)$ where $D = 1 + 16x(\lambda - x)$. For $D \in I^2$, $16x$ & $(\lambda - x)$ should not be consecutive i.e. $\lambda \neq 17x \pm 1$
Again, if $16x$ & $(\lambda - x)$ are not consecutive then $y^2 \pm y \neq 16x(\lambda - x)$ where $D = 1 + 64x(\lambda - x)$
$\Rightarrow 64x(\lambda - x)$ must be consecutive i.e. $\lambda = 65x \pm 1$ & so on.
It establishes the fact that for $\lambda = \{(4^n + 1)x \pm 1\}$ where n is odd $w = (2\lambda)^2 + 1$ will be always composite by M-product in between the common mono-wing $(2^2 + 1)$ and a general wing. For n is even, $w = (2\lambda)^2 + 1$ will be the product of two general wings i.e. by G-product when W is composite and then obviously odd element of its mono-center must be composite (because, if it is prime it fails to produce two general wings)
Now, w cannot be always composite. If w becomes always composite then the odd element of its mono-center i.e. $4xt \pm 1$ will always produce composite integer $\Rightarrow 4x \pm 1$ will always produce composite integers since t has no significance which is impossible according to Dirichlet Theorem.
Hence, $w = [2\{(4^{2n} + 1)x \pm 1\}]^2 + 1$ will produce infinitely many primes for $n, x \in I$

Corollary: $(4^{2n} + 1) \nmid (v_1 + v_2P \pm 2)$ if the prime $P = v_1v_2 \pm 1$ where $v_1, v_2 \in I_e$
For mono-center v_1v_2, P we have the composite mono-wing $= (v_2^2 + 1)(v_1^2 + P^2) = (v_1 + v_2P)^2 + 1$
$\Rightarrow v_1 + v_2P \neq 2\{(4^{2n} + 1)x \pm 1\} \Rightarrow (4^{2n} + 1) \nmid (v_1 + v_2P \pm 2)$

References:
[1] Any text book in the field of Number Theory.
[2] IJSER- Aug Edition 2013, Vol-4
[3] IJSER-Nov-Edition 2015, Vol-6

Conclusion: All the papers that I have published in this Journal are basically stages of development of a 'wing' and there lies ample scope for further development of this wing-theory to get answers of so many conjectures in Number Theory.

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